PERGAMON

International Journal of Solids and Structures 37 (2000) 1453–1469

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

A rigid line in a confocal elliptic piezoelectric inhomogeneity embedded in an infinite piezoelectric medium

Linzhi Wu*, Shanyi Du

School of Astronautics, Harbin Institute of Technology, Harbin 150001, People's Republic of China

Received 26 February 1998; in revised form 14 October 1998

Abstract

An effective method is developed and used to investigate the elastic field and electric field of a rigid line in a confocal elliptic piezoelectric inhomogeneity embedded in an infinite piezoelectric medium. The matrix is subjected to the remote antiplane shear and inplane electric field. The analytical solution is obtained using the conformal mapping and the theorem of analytic continuation. Specific solutions which are compared with existing ones are provided. The characteristics of the elastic field and electric field singularities at the rigid line tip are analyzed and the extension forces on the rigid line are derived.

1. Introduction

In studying the fracture of multi-phase piezoelectric materials with defects such as cracks, inclusions, inhomogeneities, etc., it is important to evaluate the elastic field and electric field around defects. When a rigid line is embedded in an elliptic piezoelectric inhomogeneity in an infinite piezoelectric medium, differences between the elastic, piezoelectric and dielectric constants of the elliptic inhomogeneity and matrix can cause the stress and electric displacement singularity coefficients and the corresponding extension forces on the rigid line to be greater or less than those prevailing in a homogeneous piezoelectric solid. Usually, the presence of such piezoelectric inhomogeneities plays an important role in determining the mechanical and electric behaviors of these materials (Pak, 1992).

Parton (1976) analyzed the fracture problem of piezoelectric materials using the technique of the integral equation. Deeg (1980) examined the effect of a dislocation, a crack and an inclusion upon the coupled response of piezoelectric solids. Sosa and Pak (1990) investigated the crack-tip

* Corresponding author.
E-mail address: wlz@hope.hit.edu.cn (L. Wu).

0020-7683/00/$ - see front matter
PII: S0020-7683(98)00302-3

electromechanical fields of piezoelectric solids within the realm of three-dimensional linear piezoelectricity, while Pak (1992) analyzed a piezoelectric circular inclusion problem in an infinite piezoelectric matrix. With the extended eight-dimensional formalism developed by Lothe and Barnett (1976), Kuo and Barnett (1991) and Suo et al. (1992) studied the singularities of interfacial cracks in bonded anisotropic piezoelectric media. More recently, Zhang and Tong (1996) formulated the mechanical and electric fields around an elliptic cylindrical cavity in a piezoelectric material under the remote antiplane shear and inplane electric field. Kogan et al. (1996) obtained the closed-form solutions for the stress and induction fields of a spheroidal piezoelectric inclusion in an infinite piezoelectric matrix subjected to spatially homogeneous mechanical and electric loadings at infinity. Lee and Jiang (1996) gave an exact analysis of three-dimensional piezoelectric lamina by the state space approach. Chung and Ting (1996) investigated the piezoelectric solid with an elliptic inclusion or hole using the Stroh formalism. An extensive review concerning piezoelectric materials can be found in a recent paper by Sosa and Khutoryansky (1996).

In the present paper, an effective method is developed and used to investigate the elastic field and electric field of a rigid line in a confocal elliptic piezoelectric inhomogeneity embedded in an infinite piezoelectric medium. The matrix is subjected to the remote antiplane shear and inplane electric field. The proposed method is based upon the conformal mapping and the theorem of analytic continuation. In Section 2, the basic problem is stated and the corresponding field equations and continuity conditions are outlined. In Section 3, the analytical solution for the present problem is presented according to the Laurent theorem and the reflection principle across the circle. Specific solutions which are compared with existing ones are provided. Section 4 analyzes and discusses the singularities of the stress and electric displacement fields at the rigid line tip. Furthermore, the extension forces on the rigid line are derived. Section 5 concludes this article.

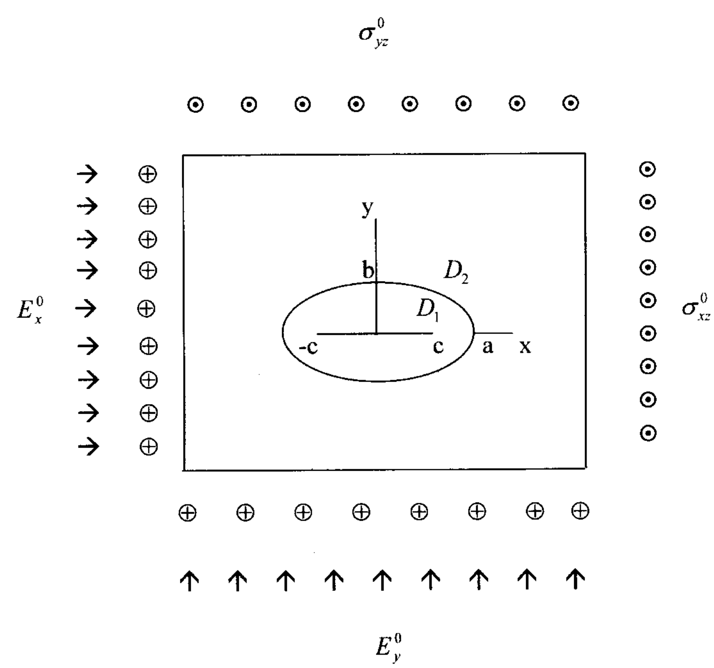

Fig. 1. A rigid line in a confocal elliptic piezoelectric inhomogeneity.

2. Statement of problem

As shown in Fig. 1, consider a rigid line in a confocal elliptic piezoelectric inhomogeneity embedded in an infinite piezoelectric matrix where the rigid line is infinitely long in the direction perpendicular to xy-plane. The inhomogeneity and matrix are assumed to have different material properties, but they have been poled along the direction perpendicular to xy-plane. The matrix, assumed to be infinite in all directions, is subjected to the far-field antiplane shear and inplane electric field.

For this problem, the displacement w is coupled with the inplane electric fields E_x and E_y. They only are the functions of coordinates x and y such as $w = w(x, y)$, $E_x = E_x(x, y)$ and $E_y = E_y(x, y)$. Referring to Pak (1990), we can write the field equations in the absence of body forces and free charges as follows

Divergence equations:

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} = 0, \quad \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} = 0 \tag{1}$$

Constitutive equations:

$$\sigma_{xz} = 2c_{44}\varepsilon_{xz} - e_{15}E_x, \quad \sigma_{yz} = 2c_{44}\varepsilon_{yz} - e_{15}E_y$$

$$D_x = 2e_{15}\varepsilon_{xz} + \kappa_{11}E_x, \quad D_y = 2e_{15}\varepsilon_{yz} + \kappa_{11}E_y \tag{2}$$

Gradient equations:

$$\varepsilon_{xz} = \frac{1}{2}\frac{\partial w}{\partial x}, \quad \varepsilon_{yz} = \frac{1}{2}\frac{\partial w}{\partial y}, \quad E_x = -\frac{\partial \varphi}{\partial x}, \quad E_y = -\frac{\partial \varphi}{\partial y} \tag{3}$$

where $(\sigma_{xz}, \sigma_{yz})$, $(\varepsilon_{xz}, \varepsilon_{yz})$, w, (D_x, D_y), (E_x, E_y) and φ are the components of stress, strain, displacement, electric displacement, electric field and electric potential, respectively. c_{44}, e_{15} and κ_{11} are the corresponding elastic, piezoelectric and dielectric constants which satisfy the relations $c_{44} > 0$ and $\kappa_{11} > 0$. Substituting eqns (2) and (3) into eqn (1) yields

$$c_{44}\nabla^2 w + e_{15}\nabla^2\varphi = 0, \quad e_{15}\nabla^2 w - \kappa_{11}\nabla^2\varphi = 0 \tag{4}$$

where ∇^2 is the two-dimensional Laplacian operator.

It is easy to show that eqn (4) can be satisfied automatically if w and φ are chosen as the real parts of the analytical functions $\Psi(z)$ and $\Phi(z)$. Thus, we have

$$w = \frac{1}{2c_{44}}\left(\Psi(z) + \overline{\Psi(z)}\right), \quad \varphi = \frac{1}{2\kappa_{11}}\left(\Phi(z) + \overline{\Phi(z)}\right) \tag{5}$$

where $z = x + iy$ is the complex variable and the overbar refers to the complex conjugate. Substituting (5) into (2) and (3), we can get the expressions of stress, electric displacement and electric fields

$$\sigma_{xz} - i\sigma_{yz} = \Psi'(z) + \frac{e_{15}}{\kappa_{11}}\Phi'(z)$$

$$D_x - iD_y = \frac{e_{15}}{c_{44}}\Psi'(z) - \Phi'(z)$$

$$E_x - iE_y = -\frac{1}{\kappa_{11}}\Phi'(z) \tag{6}$$

where prime denotes the derivatives with respect to variables. Utilizing eqn (6), the resultant force T and the resultant normal component S of the electric displacement along any arc AB can be determined by formulae

$$T=\int_A^B(\sigma_{xz}\,\mathrm{d}y-\sigma_{yz}\,\mathrm{d}x)=\frac{i}{2}\left\{\left[\overline{\Psi(z)}-\Psi(z)\right]_A^B+\frac{e_{15}}{\kappa_{11}}\left[\overline{\Phi(z)}-\Phi(z)\right]_A^B\right\}$$

$$S=\int_A^B(D_x\,\mathrm{d}y-D_y\,\mathrm{d}x)=\frac{i}{2}\left\{\frac{e_{15}}{c_{44}}\left[\overline{\Psi(z)}-\Psi(z)\right]_A^B-\left[\overline{\Phi(z)}-\Phi(z)\right]_A^B\right\} \tag{7}$$

where $[f(z)]_A^B$ represents the change of function $f(z)$ from point A to point B along the arc.

For the present problem, as shown in Fig. 2, it is convenient to map region D_2 of the z-plane into the exterior region Ω_2 of the unit circle $\Gamma_2(|\zeta|=1)$ in the ζ-plane and region D_1 into the annular region Ω_1 between Γ_2 and the circle Γ_1 of radius $1/R$ corresponding to the rigid line from $-c$ to c in the z-plane by the mapping function

$$z=\omega(\zeta)=\frac{c}{2}\left(R\zeta+\frac{1}{R\zeta}\right),\quad R\zeta=\omega^{-1}(z)=\frac{z+\sqrt{z^2-c^2}}{c} \tag{8}$$

where

$$\zeta=\xi+i\eta=\rho e^{i\theta},\quad c=(a^2-b^2)^{1/2}=a(1-\chi^2)^{1/2}$$

$$R=\left(\frac{a+b}{a-b}\right)^{1/2}=\left(\frac{1+\chi}{1-\chi}\right)^{1/2}\quad \text{with}\quad \chi=b/a \tag{9}$$

According to the mapping function (8), eqns (5)–(7) can be rewritten in the ζ-plane as follows

$$w=\frac{1}{2c_{44}}\left(\Psi(\zeta)+\overline{\Psi(\zeta)}\right),$$

$$\varphi=\frac{1}{2\kappa_{11}}\left(\Phi(\zeta)+\overline{\Phi(\zeta)}\right); \tag{10}$$

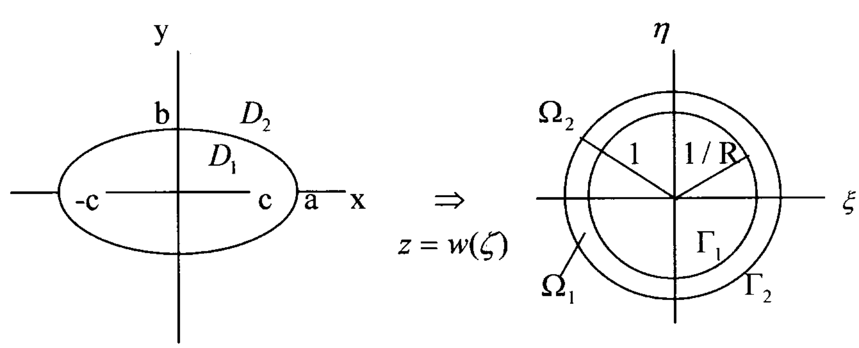

Fig. 2. A schematic of conformal mapping.

$$\sigma_{xz} - i\sigma_{yz} = \frac{\Psi'(\zeta)}{\omega'(\zeta)} + \frac{e_{15}}{\kappa_{11}}\frac{\Phi'(\zeta)}{\omega'(\zeta)},$$

$$D_x - iD_y = \frac{e_{15}}{c_{44}}\frac{\Psi'(\zeta)}{\omega'(\zeta)} - \frac{\Phi'(\zeta)}{\omega'(\zeta)},$$

$$E_x - iE_y = -\frac{1}{\kappa_{11}}\frac{\Phi'(\zeta)}{\omega'(\zeta)}; \tag{11}$$

$$iT = \frac{1}{2}\left\{\left[\Psi(\zeta) - \overline{\Psi(\zeta)}\right]_A^B + \frac{e_{15}}{\kappa_{11}}\left[\Phi(\zeta) - \overline{\Phi(\zeta)}\right]_A^B\right\},$$

$$iS = \frac{1}{2}\left\{\frac{e_{15}}{c_{44}}\left[\Psi(\zeta) - \overline{\Psi(\zeta)}\right]_A^B - \left[\Phi(\zeta) - \overline{\Phi(\zeta)}\right]_A^B\right\} \tag{12}$$

where $\Psi(\zeta) = \Psi[\omega(\zeta)]$ and $\Phi(\zeta) = \Phi[\omega(\zeta)]$. Evidently, if the complex functions $\Psi(\zeta)$ and $\Phi(\zeta)$ in regions Ω_1 and Ω_2 are determined, we can obtain the corresponding elastic field and electric field. For this purpose, we first give the continuity conditions along the interface which is assumed to be perfectly bonded and the boundary conditions. The assumption of perfect bonding and that of no free charges and forces along the interface between regions Ω_1 and Ω_2 imply the continuity of displacement, electric potential, traction and normal component of the electric displacement on Γ_2. These conditions can be expressed as

$$\mu_1\left[\Psi_1(\zeta) + \overline{\Psi_1(\zeta)}\right] = \Psi_2(\zeta) + \overline{\Psi_2(\zeta)}$$

$$\mu_2\left[\Phi_1(\zeta) + \overline{\Phi_1(\zeta)}\right] = \Phi_2(\zeta) + \overline{\Phi_2(\zeta)} \tag{13}$$

$$\Psi_1(\zeta) - \overline{\Psi_1}(\zeta) + \alpha_1\left[\Phi_1(\zeta) - \overline{\Phi_1(\zeta)}\right] = \Psi_2(\zeta) - \overline{\Psi_2(\zeta)} + \alpha_2\left[\Phi_2(\zeta) - \overline{\Phi_2(\zeta)}\right]$$

$$\beta_1\left[\Psi_1(\zeta) - \overline{\Psi_1(\zeta)}\right] - \Phi_1(\zeta) + \overline{\Phi_1(\zeta)} = \beta_2\left[\Psi_2(\zeta) - \overline{\Psi_2(\zeta)}\right] - \Phi_2(\zeta) + \overline{\Phi_2(\zeta)} \tag{14}$$

with

$$\mu_1 = c_{44}^2/c_{44}^1, \quad \mu_2 = \kappa_{11}^2/\kappa_{11}^1, \quad \alpha_1 = e_{15}^1/\kappa_{11}^1, \quad \alpha_2 = e_{15}^2/\kappa_{11}^2,$$

$$\beta_1 = e_{15}^1/c_{44}^1, \quad \beta_2 = e_{15}^2/c_{44}^2 \tag{15}$$

where superscripts 1 and 2 of material constants (c_{44}, e_{15}, κ_{11}) and subscripts 1 and 2 of potential functions (Ψ, Φ) correspond to the inhomogeneity Ω_1 and matrix Ω_2, respectively. In this paper, the conductive rigid line and the insulating one are considered, respectively. According to eqn (12), the traction and normal component of electric displacement on Γ_1 can be expressed as

$$\Psi_1(\zeta) + \overline{\Psi_1(\zeta)} = 0 \quad |\zeta| = 1/R \tag{16}$$

$$\Phi_1(\zeta) + \overline{\Phi_1(\zeta)} = 0 \quad |\zeta| = 1/R \tag{17}$$

for the conductive rigid line;

$$\Psi_1(\zeta) + \overline{\Psi_1(\zeta)} = 0 \quad |\zeta| = 1/R \tag{18}$$

$$\beta_1\left[\Psi_1(\zeta) - \overline{\Psi_1(\zeta)}\right] - \Phi_1(\zeta) + \overline{\Phi_1(\zeta)} = 0 \quad |\zeta| = 1/R \tag{19}$$

for the insulating rigid line. Since the solving process of these two problems is similar, we will confine our attention only to the problem of the conductive rigid line. For the insulating rigid line, only the corresponding results are given.

For the boundary conditions, we have in terms of eqns (8) and (11)

$$E_x^0 - iE_y^0 = \lim_{|\zeta|\to\infty}\left(E_x - iE_y\right) = -\frac{2}{Rc\kappa_{11}^2}\lim_{|\zeta|\to\infty}\Phi_2'(\zeta) \tag{20}$$

$$\sigma_{xz}^0 - i\sigma_{yz}^0 = \lim_{|\zeta|\to\infty}\left(\sigma_{xz} - i\sigma_{yz}\right) = \frac{2}{Rc}\lim_{|\zeta|\to\infty}\Psi_2'(\zeta) + \frac{2\alpha_2}{Rc}\lim_{|\zeta|\to\infty}\Phi_2'(\zeta) \tag{21}$$

where $(\sigma_{xz}^0, \sigma_{yz}^0)$ and (E_x^0, E_y^0) are the uniform shear stresses and electric fields at infinity.

3. Solution to elastic and electric fields

In this section, the theorem of analytic continuation is used to solve the present problem. Following eqns (20) and (21), we know that both $\Psi_2(\zeta)$ and $\Phi_2(\zeta)$ tend towards infinity when $|\zeta|$ tends to infinity. Thus, we can set $\Psi_2(\zeta) = \psi(\zeta)\zeta$ and $\Phi_2(\zeta) = \phi(\zeta)\zeta$. Since $\Psi_2(\zeta)$ and $\Phi_2(\zeta)$ are analytic in region Ω_2, $\psi(\zeta)$ and $\phi(\zeta)$ also are analytic in Ω_2. Eliminating functions $\overline{\Psi_2(\zeta)}$ and $\overline{\Phi_2(\zeta)}$ from eqns (13) and (14), we have

$$\begin{gathered}\left(1+\mu_1\right)\Psi_1(\zeta) - \left(1-\mu_1\right)\overline{\Psi_1(\zeta)} + (\alpha_1+\alpha_2\mu_2)\Phi_1(\zeta) - (\alpha_1-\alpha_2\mu_2)\overline{\Phi_1(\zeta)} = 2\Psi_2(\zeta) + 2\alpha_2\Phi_2(\zeta)\\ \left(\beta_1+\beta_2\mu_1\right)\Psi_1(\zeta) - \left(\beta_1-\beta_2\mu_1\right)\overline{\Psi_1(\zeta)} - \left(1+\mu_2\right)\Phi_1(\zeta) + \left(1-\mu_2\right)\overline{\Phi_1(\zeta)} = 2\beta_2\Psi_2(\zeta) - 2\Phi_2(\zeta) \quad |\zeta| = 1\end{gathered} \tag{22}$$

Substituting $\Psi_2(\zeta) = \psi(\zeta)\zeta$ and $\Phi_2(\zeta) = \phi(\zeta)\zeta$ into eqn (22), we can obtain

$$\begin{gathered}\left(1+\mu_1\right)\frac{\Psi_1(\zeta)}{\zeta} - \left(1-\mu_1\right)\frac{\overline{\Psi_1(\zeta)}}{\zeta} + (\alpha_1+\alpha_2\mu_2)\frac{\Phi_1(\zeta)}{\zeta} - (\alpha_1-\alpha_2\mu_2)\frac{\overline{\Phi_1(\zeta)}}{\zeta} = 2\psi(\zeta) + 2\alpha_2\phi(\zeta)\\ \left(\beta_1+\beta_2\mu_1\right)\frac{\Psi_1(\zeta)}{\zeta} - \left(\beta_1-\beta_2\mu_1\right)\frac{\overline{\Psi_1(\zeta)}}{\zeta} - \left(1+\mu_2\right)\frac{\Phi_1(\zeta)}{\zeta} + \left(1-\mu_2\right)\frac{\overline{\Phi_1(\zeta)}}{\zeta} = 2\beta_2\psi(\zeta) - 2\phi(\zeta) \quad |\zeta| = 1\end{gathered} \tag{23}$$

According to the Laurent theorem, functions $\Psi_1(\zeta)$ and $\Phi_1(\zeta)$, analytic in the ring Ω_1, can be expressed as

$$\Psi_1(\zeta) = \Psi_1^+(\zeta) + \Psi_1^-(\zeta), \quad \Phi_1(\zeta) = \Phi_1^+(\zeta) + \Phi_1^-(\zeta) \quad \zeta \in \Omega_1 \tag{24}$$

where $\Psi_1^+(\zeta)$ and $\Phi_1^+(\zeta)$ are analytic in region $|\zeta|<1$ and $\Psi_1^-(\zeta)$ and $\Phi_1^-(\zeta)$ are analytic in region

$|\zeta| > 1/R$. Substituting eqn (24) into eqns (16), (17) and (23), we have

$$\Psi_1^+(\zeta) + \overline{\Psi_1^-(\zeta)} = -\Psi_1^-(\zeta) - \overline{\Psi_1^+(\zeta)}$$

$$\Phi_1^+(\zeta) + \overline{\Phi_1^-(\zeta)} = -\Phi_1^-(\zeta) - \overline{\Phi_1^+(\zeta)} \quad |\zeta| = 1/R \tag{25}$$

$$\left(1+\mu_1\right)\frac{\Psi_1^+(\zeta)}{\zeta} - \left(1-\mu_1\right)\frac{\overline{\Psi_1^-(\zeta)}}{\zeta} + (\alpha_1+\alpha_2\mu_2)\frac{\Phi_1^+(\zeta)}{\zeta} - (\alpha_1-\alpha_2\mu_2)\frac{\overline{\Phi_1^-(\zeta)}}{\zeta}$$

$$= -\left(1+\mu_1\right)\frac{\Psi_1^-(\zeta)}{\zeta} + \left(1-\mu_1\right)\frac{\overline{\Psi_1^+(\zeta)}}{\zeta} - (\alpha_1+\alpha_2\mu_2)\frac{\Phi_1^-(\zeta)}{\zeta}$$

$$+(\alpha_1-\alpha_2\mu_2)\frac{\overline{\Phi_1^+(\zeta)}}{\zeta} + 2\psi(\zeta) + 2\alpha_2\phi(\zeta)$$

$$\left(\beta_1+\beta_2\mu_1\right)\frac{\Psi_1^+(\zeta)}{\zeta} - \left(\beta_1-\beta_2\mu_1\right)\frac{\overline{\Psi_1^-(\zeta)}}{\zeta} - \left(1+\mu_2\right)\frac{\Phi_1^+(\zeta)}{\zeta} + \left(1-\mu_2\right)\frac{\overline{\Phi_1^-(\zeta)}}{\zeta}$$

$$= -\left(\beta_1+\beta_2\mu_1\right)\frac{\Psi_1^-(\zeta)}{\zeta} + \left(\beta_1-\beta_2\mu_1\right)\frac{\overline{\Psi_1^+(\zeta)}}{\zeta} + \left(1+\mu_2\right)\frac{\Phi_1^-(\zeta)}{\zeta} - \left(1-\mu_2\right)\frac{\overline{\Phi_1^+(\zeta)}}{\zeta}$$

$$+2\beta_2\psi(\zeta) - 2\phi(\zeta) \quad |\zeta| = 1 \tag{26}$$

Application of the theorem of analytic continuation to eqns (25) and (26) gives the possibility to introduce the four analytic functions

$$F_1(\zeta) = \begin{cases} \Psi_1^+(\zeta) + \overline{\Psi_1^-\left(\dfrac{1}{R^2\bar{\zeta}}\right)} & |\zeta| < \dfrac{1}{R} \\ -\Psi_1^-(\zeta) - \overline{\Psi_1^+\left(\dfrac{1}{R^2\bar{\zeta}}\right)} & |\zeta| > \dfrac{1}{R} \end{cases} \tag{27}$$

$$F_2(\zeta) = \begin{cases} \Phi_1^+(\zeta) + \overline{\Phi_1^-\left(\dfrac{1}{R^2\bar{\zeta}}\right)} & |\zeta| < \dfrac{1}{R} \\ -\Phi_1^-(\zeta) - \overline{\Phi_1^+\left(\dfrac{1}{R^2\bar{\zeta}}\right)} & |\zeta| > \dfrac{1}{R} \end{cases} \tag{28}$$

$$F_3(\zeta)=\begin{cases}(1+\mu_1)\dfrac{\Psi_1^+(\zeta)}{\zeta}-(1-\mu_1)\dfrac{\overline{\Psi_1^-(1/\bar{\zeta})}}{\zeta}+(\alpha_1+\alpha_2\mu_2)\dfrac{\Phi_1^+(\zeta)}{\zeta} & \\ \quad -(\alpha_1-\alpha_2\mu_2)\dfrac{\overline{\Phi_1^-(1/\bar{\zeta})}}{\zeta} & |\zeta|<1 \\ -(1+\mu_1)\dfrac{\Psi_1^-(\zeta)}{\zeta}+(1-\mu_1)\dfrac{\overline{\Psi_1^+(1/\bar{\zeta})}}{\zeta}-(\alpha_1+\alpha_2\mu_2)\dfrac{\Phi_1^-(\zeta)}{\zeta} & \\ \quad +(\alpha_1-\alpha_2\mu_2)\dfrac{\overline{\Phi_1^+(1/\bar{\zeta})}}{\zeta}+2\psi(\zeta)+2\alpha_2\phi(\zeta) & |\zeta|>1\end{cases} \tag{29}$$

$$F_4(\zeta)=\begin{cases}(\beta_1+\beta_2\mu_1)\dfrac{\Psi_1^+(\zeta)}{\zeta}-(\beta_1-\beta_2\mu_1)\dfrac{\overline{\Psi_1^-(1/\bar{\zeta})}}{\zeta}-(1+\mu_2)\dfrac{\Phi_1^+(\zeta)}{\zeta} & \\ \quad +(1-\mu_2)\dfrac{\overline{\Phi_1^-(1/\bar{\zeta})}}{\zeta} & |\zeta|<1 \\ -(\beta_1+\beta_2\mu_1)\dfrac{\Psi_1^-(\zeta)}{\zeta}+(\beta_1-\beta_2\mu_1)\dfrac{\overline{\Psi_1^+(1/\bar{\zeta})}}{\zeta}+(1+\mu_2)\dfrac{\Phi_1^-(\zeta)}{\zeta} & \\ \quad -(1-\mu_2)\dfrac{\overline{\Phi_1^+(1/\bar{\zeta})}}{\zeta}+2\beta_2\psi(\zeta)-2\phi(\zeta) & |\zeta|>1\end{cases} \tag{30}$$

By the Liouville theorem, we have

$$F_1(\zeta)=C_1,\quad F_2(\zeta)=C_2,\quad F_3(\zeta)=C_3,\quad F_4(\zeta)=C_4 \tag{31}$$

where C_1, C_2, C_3 and C_4 are the complex constants which can be determined from eqns (29) and (30) by the use of (20) and (21). Thus, we have

$$\Psi_1^+(\zeta)+\overline{\Psi_1^-\left(\frac{1}{R^2\bar{\zeta}}\right)}=C_1\quad |\zeta|<\frac{1}{R} \tag{32}$$

$$\Phi_1^+(\zeta)+\overline{\Phi_1^-\left(\frac{1}{R^2\bar{\zeta}}\right)}=C_2\quad |\zeta|<\frac{1}{R} \tag{33}$$

$$(1+\mu_1)\Psi_1^+(\zeta)-(1-\mu_1)\overline{\Psi_1^-\left(\frac{1}{\bar{\zeta}}\right)}+(\alpha_1+\alpha_2\mu_2)\Phi_1^+(\zeta)-(\alpha_1-\alpha_2\mu_2)\overline{\Phi_1^-\left(\frac{1}{\bar{\zeta}}\right)}=C_3\zeta\quad |\zeta|<1 \tag{34}$$

$$(\beta_1+\beta_2\mu_1)\Psi_1^+(\zeta)-(\beta_1-\beta_2\mu_1)\overline{\Psi_1^-\left(\frac{1}{\bar{\zeta}}\right)}-(1+\mu_2)\Phi_1^+(\zeta)+(1-\mu_2)\overline{\Phi_1^-\left(\frac{1}{\bar{\zeta}}\right)}=C_4\zeta\quad |\zeta|<1 \tag{35}$$

According to the reflection principle across circle $|\zeta|=1/R$ and eqn (31), we find that only when constants C_1 and C_2 are zeroes or imaginary numbers, eqns (27) and (28) are compatible with eqns (32) and (33), respectively. In Section 4, we will show that constants C_1 and C_2 really represent the rigid body motion and equipotential field which can be ignored. According to the principle of analytic continuation, eqns (32)–(35) can be solved in region $|\zeta|<1/R$. Eliminating functions $\Psi_1^+(\zeta)$ and $\Phi_1^+(\zeta)$ from eqns (32)–(35), we have after some manipulation

$$\begin{bmatrix} \overline{\Psi_1^-\left(\dfrac{1}{R^2\bar{\zeta}}\right)} \\ \overline{\Phi_1^-\left(\dfrac{1}{R^2\bar{\zeta}}\right)} \end{bmatrix} = -\frac{1}{A}\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}\begin{bmatrix} \overline{\Psi_1^-\left(\dfrac{1}{\bar{\zeta}}\right)} \\ \overline{\Phi_1^-\left(\dfrac{1}{\bar{\zeta}}\right)} \end{bmatrix} - \frac{1}{A}\begin{bmatrix} 1+\mu_2 & \alpha_1+\alpha_2\mu_2 \\ \beta_1+\beta_2\mu_1 & -1-\mu_1 \end{bmatrix}\begin{bmatrix} C_3 \\ C_4 \end{bmatrix}\zeta + \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

$$|\zeta|<\frac{1}{R} \tag{36}$$

where

$$A = (1+\mu_1)(1+\mu_2) + (\alpha_1+\alpha_2\mu_2)(\beta_1+\beta_2\mu_1)$$

$$A_1 = (1-\mu_1)(1+\mu_2) + (\alpha_1+\alpha_2\mu_2)(\beta_1-\beta_2\mu_1)$$

$$A_2 = 2\mu_2(\alpha_1-\alpha_2)$$

$$A_3 = 2\mu_1(\beta_2-\beta_1)$$

$$A_4 = (1+\mu_1)(1-\mu_2) + (\alpha_1-\alpha_2\mu_2)(\beta_1+\beta_2\mu_1) \tag{37}$$

According to the reflection principle across circle $|\zeta| = 1/R$, eqn (36) can further be written as

$$\begin{bmatrix} \Psi_1^-(\zeta) \\ \Phi_1^-(\zeta) \end{bmatrix} = \begin{bmatrix} \overline{C_1} \\ \overline{C_2} \end{bmatrix} - \frac{1}{A}\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}\begin{bmatrix} \Psi_1^-(R^2\zeta) \\ \Phi_1^-(R^2\zeta) \end{bmatrix} - \frac{1}{A}\begin{bmatrix} 1+\mu_2 & \alpha_1+\alpha_2\mu_2 \\ \beta_1+\beta_2\mu_1 & -1-\mu_1 \end{bmatrix}\begin{bmatrix} \overline{C_3} \\ \overline{C_4} \end{bmatrix}\frac{1}{R^2\zeta} \quad |\zeta|>\frac{1}{R} \tag{38}$$

where constants C_1 and C_2 are considered as zeroes or imaginary numbers.

Equation (38) is, in fact, the functional equation to determine the unknown functions $\Psi_1^-(\zeta)$ and $\Phi_1^-(\zeta)$. To solve these two functions, the sequential transformations ($|\zeta| > 1/R$) can be adopted. However, it is easily found that the solution to eqn (38) can be expressed as

$$\begin{bmatrix} \Psi_1^-(\zeta) \\ \Phi_1^-(\zeta) \end{bmatrix} = -\left\{AR^2\mathbf{I} + \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}\right\}^{-1}\begin{bmatrix} 1+\mu_2 & \alpha_1+\alpha_2\mu_2 \\ \beta_1+\beta_2\mu_1 & -1-\mu_1 \end{bmatrix}\begin{bmatrix} \overline{C_3} \\ \overline{C_4} \end{bmatrix}\frac{1}{\zeta}$$

$$+ A\left\{A\mathbf{I} + \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}\right\}^{-1}\begin{bmatrix} \overline{C_1} \\ \overline{C_2} \end{bmatrix} \quad |\zeta| > \frac{1}{R} \tag{39}$$

where $\mathbf{I}$ denotes the second-order unit matrix. In the Appendix, we will prove that the matrices in eqn (39) are inverse. Obviously, constants C_1 and C_2 represent the rigid body motion and the equipotential field, respectively. They can be taken as zeroes. Substituting (39) into (34) and (35), we can obtain after some manipulation

$$\begin{bmatrix} \Psi_1^+(\zeta) \\ \Phi_1^+(\zeta) \end{bmatrix} = R^2 \left\{ AR^2\mathbf{I} + \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \right\}^{-1} \begin{bmatrix} 1+\mu_2 & \alpha_1+\alpha_2\mu_2 \\ \beta_1+\beta_2\mu_1 & -1-\mu_1 \end{bmatrix} \begin{bmatrix} C_3 \\ C_4 \end{bmatrix} \zeta \quad |\zeta|<1 \tag{40}$$

Subsequently, we determine the functions $\Psi_1(\zeta)$, $\Phi_1(\zeta)$, $\Psi_2(\zeta)$ and $\Phi_2(\zeta)$. Substituting eqns (39) and (40) into eqns (24) and (29)–(31) and according to the relations $\Psi_2(\zeta) = \psi(\zeta)\zeta$ and $\Phi_2(\zeta) = \phi(\zeta)\zeta$, we can obtain

$$\begin{bmatrix} \Psi_1(\zeta) \\ \Phi_1(\zeta) \end{bmatrix} = \left\{ AR^2\mathbf{I} + \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \right\}^{-1} \begin{bmatrix} 1+\mu_2 & \alpha_1+\alpha_2\mu_2 \\ \beta_1+\beta_2\mu_1 & -1-\mu_1 \end{bmatrix} \left\{ \begin{bmatrix} C_3 \\ C_4 \end{bmatrix} R^2\zeta - \begin{bmatrix} \overline{C_3} \\ \overline{C_4} \end{bmatrix} \frac{1}{\zeta} \right\} \quad \frac{1}{R} < |\zeta| < 1 \tag{41}$$

$$\begin{bmatrix} \mu_2\Psi_2(\zeta) \\ \mu_1\Phi_2(\zeta) \end{bmatrix} = \left\{ R^2\mu_1\mu_2\mathbf{I} - \frac{1+R^2}{4(1+\alpha_2\beta_2)} \begin{bmatrix} A-A_4 & A_2 \\ A_3 & A-A_1 \end{bmatrix} \right\} \left\{ AR^2\mathbf{I} + \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \right\}^{-1}$$

$$\times \begin{bmatrix} 1+\mu_2 & \alpha_1+\alpha_2\mu_2 \\ \beta_1+\beta_2\mu_1 & -1-\mu_1 \end{bmatrix} \begin{bmatrix} \overline{C_3} \\ \overline{C_4} \end{bmatrix} \frac{1}{\zeta} + \frac{1}{2(1+\alpha_2\beta_2)} \begin{bmatrix} (C_3+\alpha_2C_4)\mu_2 \\ (C_3\beta_2-C_4)\mu_1 \end{bmatrix} \zeta \quad |\zeta|>1 \tag{42}$$

For constants C_3 and C_4, we can obtain by substituting eqn (42) into eqns (20) and (21)

$$\begin{bmatrix} C_3 \\ C_4 \end{bmatrix} = Rc \begin{bmatrix} 1 & 0 \\ \beta_2 & (1+\alpha_2\beta_2)\kappa_{11}^2 \end{bmatrix} \begin{bmatrix} \sigma_{xz}^0 - i\sigma_{yz}^0 \\ E_x^0 - iE_y^0 \end{bmatrix} \tag{43}$$

Substituting (43) into (41) and (42), we determine the corresponding analytical functions $\Psi_1(\zeta)$, $\Phi_1(\zeta)$, $\Psi_2(\zeta)$ and $\Phi_2(\zeta)$. Furthermore, we can solve the elastic field and electric field from eqns (10) and (11). To analyze the elastic field and electric field around the rigid line tip, we give the expressions of the stress field and electric displacement field in Ω_1 below. Substituting eqn (41) into eqns (10) and (11), we can obtain after some manipulation

$$\begin{bmatrix} D_x - iD_y \\ \sigma_{xz} - i\sigma_{yz} \end{bmatrix} = \frac{2R\zeta^2}{c(R^2\zeta^2-1)} \begin{bmatrix} \beta_1 & -1 \\ 1 & \alpha_1 \end{bmatrix} \left\{ AR^2\mathbf{I} + \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \right\}^{-1}$$

$$\times \begin{bmatrix} 1+\mu_2 & \alpha_1+\alpha_2\mu_2 \\ \beta_1+\beta_2\mu_1 & -1-\mu_1 \end{bmatrix} \left\{ R^2 \begin{bmatrix} C_3 \\ C_4 \end{bmatrix} + \begin{bmatrix} \overline{C_3} \\ \overline{C_4} \end{bmatrix} \frac{1}{\zeta^2} \right\} \quad \zeta \in \Omega_1 \tag{44}$$

To determine the elastic field and electric field in the z-plane, we use the inverse transformation $\omega^{-1}(z)$ of (8). Since we are interested in the field near the rigid line tip, it is convenient to choose a local polar coordinate system (r, θ) with the origin at $x = c$, i.e.,

$$z = c + re^{i\theta} \tag{45}$$

Substituting eqn (45) into the second relation of eqn (8) and assuming $r \ll c$ yield

$$R\zeta = 1 + \left(\frac{2r}{c}\right)^{1/2} e^{i\theta/2} + \frac{r}{c} e^{i\theta} \tag{46}$$

where the terms of $(r/c)^{3/2}$ and higher have been dropped. Thus, we can obtain the stress field and electric displacement field around the rigid line tip by substitution of eqn (46) into (44) and the use of eqn (43)

$$\begin{bmatrix} D_x - iD_y \\ \sigma_{xz} - i\sigma_{yz} \end{bmatrix} = 2R^2 \left(\frac{c}{2r}\right)^{1/2} e^{-i\theta/2} \begin{bmatrix} \beta_1 & -1 \\ 1 & \alpha_1 \end{bmatrix} \left\{ AR^2\mathbf{I} + \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \right\}^{-1}$$

$$\times \begin{bmatrix} 1+\mu_2 & \alpha_1 + \alpha_2\mu_2 \\ \beta_1 + \beta_2\mu_1 & -1-\mu_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \beta_2 & (1+\alpha_2\beta_2)\kappa_{11}^2 \end{bmatrix} \begin{bmatrix} \sigma_{xz}^0 \\ E_x^0 \end{bmatrix} \quad \zeta \in \Omega_1 \tag{47}$$

Equation (47) corresponds to the case of the conductive rigid line. For the insulating rigid line, a similar derivation can be performed. The corresponding expression is written as

$$\begin{bmatrix} D_x - iD_y \\ \sigma_{xz} - i\sigma_{yz} \end{bmatrix} = \frac{1}{Rc}\left(\frac{c}{2r}\right)^{1/2} e^{-i\theta/2} \begin{bmatrix} \beta_1 & -1 \\ 1 & \alpha_1 \end{bmatrix} \left\{ R^2 \left[AR^2\mathbf{I} + \begin{pmatrix} A_1 & A_2 \\ A_3' & A_4' \end{pmatrix} \right]^{-1} \begin{bmatrix} B_1 \\ B_2' \end{bmatrix} + \frac{1}{A} \begin{bmatrix} \overline{B_1} \\ \overline{B_2} \end{bmatrix} \right.$$

$$\left. - \frac{1}{A} \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \left[AR^2\mathbf{I} + \begin{pmatrix} A_1 & A_2 \\ A_3' & A_4' \end{pmatrix} \right]^{-1} \begin{bmatrix} \overline{B_1} \\ \overline{B}_2' \end{bmatrix} \right\} \quad \zeta \in \Omega_1 \tag{48}$$

where

$$A_3' = 2\left[\beta_1 - \beta_2\mu_1 + \beta_1\mu_2 - \beta_1\mu_1\mu_2 + \alpha_1\beta_1^2 - \alpha_2\beta_1\beta_2\mu_1\mu_2\right]$$

$$A_4' = -(1+\mu_1)(1-\mu_2) - \alpha_1\beta_1 + \alpha_2\beta_2\mu_1\mu_2 - 4\alpha_2\beta_1\mu_2 + 4\alpha_1\beta_1\mu_2$$

$$\begin{bmatrix} \overline{B_1} \\ \overline{B_2} \\ \overline{B}_2' \end{bmatrix} = \begin{bmatrix} 1+\mu_2 & \alpha_1 + \alpha_2\mu_2 \\ \beta_1 + \beta_2\mu_1 & -1-\mu_1 \\ \beta_1 + 2\beta_1\mu_2 - \beta_2\mu_1 & 1 + \mu_1 + 2\alpha_1\beta_1 + 2\alpha_2\beta_1\mu_2 \end{bmatrix} \begin{bmatrix} C_3 \\ C_4 \end{bmatrix} \tag{49}$$

Below, we consider two specific cases. When the piezoelectric coupling effect is absent or $e_{15}^1 = 0$ and $e_{15}^2 = 0$, we have in terms of eqns (15), (37) and (49)

$$\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0, \quad A_1 = (1-\mu_1)(1+\mu_2), \quad A_2 = A_3 = A_3' = 0$$

$$A_4 = -A_4' = (1+\mu_1)(1-\mu_2), \quad A = (1+\mu_1)(1+\mu_2)$$

$$\overline{B_1} = (1+\mu_2)C_3, \quad \overline{B_2} = -\overline{B}_2' = -(1+\mu_1)C_4 \tag{50}$$

Substituting eqn (50) into eqns (47) and (48), we can get

$$\begin{bmatrix} D_x - iD_y \\ \sigma_{xz} - i\sigma_{yz} \end{bmatrix} = 2R^2\left(\frac{c}{2r}\right)^{1/2} \mathrm{e}^{-i\theta/2} \begin{bmatrix} \dfrac{\kappa_{11}^2 E_x^0}{(1+\mu_2)R^2 + 1 - \mu_2} \\ \dfrac{\sigma_{xz}^0}{(1+\mu_1)R^2 + 1 - \mu_1} \end{bmatrix} \tag{51}$$

for the conductive rigid line;

$$\begin{bmatrix} D_x - iD_y \\ \sigma_{xz} - i\sigma_{yz} \end{bmatrix} = 2R^2\left(\frac{c}{2r}\right)^{1/2} \mathrm{e}^{-i\theta/2} \begin{bmatrix} \dfrac{-i\kappa_{11}^2 E_y^0}{(1+\mu_2)R^2 - (1 - \mu_2)} \\ \dfrac{\sigma_{xz}^0}{(1+\mu_1)R^2 + 1 - \mu_1} \end{bmatrix} \tag{52}$$

for the insulating rigid line. From (51) and (52), it can be found that the elastic field at the conductive rigid line tip agrees with one at the insulating rigid line tip.

When the inhomogeneity and matrix have the same elastic, piezoelectric and dielectric constants, we have the following relations

$$\mu_1 = \mu_2 = 1, \quad \alpha_1 = \alpha_2 = \alpha, \quad \beta_1 = \beta_2 = \beta,$$

$$A = 4(1 + \alpha\beta), \quad A_1 = A_2 = A_3 = A_3' = A_4 = A_4' = 0$$

$$\overline{B_1} = 2C_3 + 2\alpha C_4, \quad \overline{B_2} = 2\beta C_3 - 2C_4, \quad \overline{B}_2' = 2\beta C_3 + 2(1 + 2\alpha\beta)C_4 \tag{53}$$

Substituting (53) into (47) and (48), we have

$$\begin{bmatrix} D_x - iD_y \\ \sigma_{xz} - i\sigma_{yz} \end{bmatrix} = \left(\frac{c}{2r}\right)^{1/2} \mathrm{e}^{-i\theta/2} \begin{bmatrix} \beta\sigma_{xz}^0 + (1+\alpha\beta)\kappa_{11}E_x^0 \\ \sigma_{xz}^0 \end{bmatrix} \tag{54}$$

for the conductive rigid line;

$$\begin{bmatrix} D_x - iD_y \\ \sigma_{xz} - i\sigma_{yz} \end{bmatrix} = \left(\frac{c}{2r}\right)^{1/2} \mathrm{e}^{-i\theta/2} \begin{bmatrix} -i\left[\beta\sigma_{yz}^0 + (1+\alpha\beta)\kappa_{11}E_y^0\right] \\ (1+\alpha\beta)\left(\sigma_{xz}^0 + \alpha\kappa_{11}E_x^0\right) + i\alpha\left[\beta\sigma_{yz}^0 + (1+\alpha\beta)\kappa_{11}E_y^0\right] \end{bmatrix} \tag{55}$$

for the insulating rigid line. It is easily seen that when $e_{15}^1 = 0$ and $e_{15}^2 = 0$, the corresponding elastic field agrees with the expressions of Wang et al. (1986). In the next section, we will derive the stress and electric displacement singularity coefficients and extension forces on the rigid line according to the expressions obtained above.

4. Results and discussions

From eqns (47) and (48), it is clear that the stress field and electric displacement field have the singularity at the rigid line tip. For the conductive rigid line, the singularity is caused only by the shear stress σ_{xz}^0 and electric field E_x^0. For the insulating rigid line, however, the shear stresses (σ_{xz}^0, σ_{yz}^0) and electric fields (E_x^0, E_y^0) have the effect on the singularity of the stress and electric displacement fields. According to Pak (1990) and eqns (47) and (48), the stress and electric displacement singularity coefficients which are similarly defined in Wang et al. (1986) are written as

$$\begin{bmatrix} K_x^d - iK_y^d \\ K_{3x}^\sigma - iK_{3y}^\sigma \end{bmatrix} = \lim_{r\to 0}(2\pi r)^{1/2}\begin{bmatrix} D_x - iD_y \\ \sigma_{xz} - i\sigma_{yz} \end{bmatrix}_{\theta=0}$$

$$= 2R^2(c\pi)^{1/2}\begin{bmatrix} \beta_1 & -1 \\ 1 & \alpha_1 \end{bmatrix}\left\{AR^2\mathbf{I} + \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}\right\}^{-1}\begin{bmatrix} 1+\mu_2 & \alpha_1+\alpha_2\mu_2 \\ \beta_1+\beta_2\mu_1 & -1-\mu_1 \end{bmatrix}$$

$$\times\begin{bmatrix} 1 & 0 \\ \beta_2 & (1+\alpha_2\beta_2)\kappa_{11}^2 \end{bmatrix}\begin{bmatrix} \sigma_{xz}^0 \\ E_x^0 \end{bmatrix} \tag{56}$$

for the conductive rigid line;

$$\begin{bmatrix} K_x^d - iK_y^d \\ K_{3x}^\sigma - iK_{3y}^\sigma \end{bmatrix} = \lim_{r\to 0}(2\pi r)^{1/2}\begin{bmatrix} D_x - iD_y \\ \sigma_{xz} - i\sigma_{yz} \end{bmatrix}_{\theta=0}$$

$$= \frac{(c\pi)^{1/2}}{Rc}\begin{bmatrix} \beta_1 & -1 \\ 1 & \alpha_1 \end{bmatrix}\left\{R^2\left[AR^2\mathbf{I} + \begin{pmatrix} A_1 & A_2 \\ A_3' & A_4' \end{pmatrix}\right]^{-1}\begin{bmatrix} B_1 \\ B_2' \end{bmatrix} + \frac{1}{A}\begin{bmatrix} \overline{B_1} \\ \overline{B_2} \end{bmatrix}\right.$$
$$\left. - \frac{1}{A}\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}\left[AR^2\mathbf{I} + \begin{pmatrix} A_1 & A_2 \\ A_3' & A_4' \end{pmatrix}\right]^{-1}\begin{bmatrix} \overline{B_1} \\ \overline{B_2'} \end{bmatrix}\right\} \tag{57}$$

for the insulating rigid line. From eqn (56), it can be found that only the stress singularity coefficient K_{3x}^σ and electric displacement singularity coefficient K_x^d are nonzero constants when the rigid line is conductive. However, when the rigid line is insulating, all the stress and electric displacement singularity coefficients are nonzero constants. It is easily verified that when the inhomogeneity and matrix have the same elastic, piezoelectric and dielectric constants and the piezoelectric coupling effect is absent, the present elastic field can degenerate into the result of Wang et al. (1986).

Subsequently, we determine the path-independent integral similar to the J integral in the elastic problem or the extension force on the rigid line. According to Rice (1968, 1969) and Pak (1990), the path-independent integral or the extension force on the rigid line can be expressed as

$$J_p = \int_\Gamma \left(Hn_x - \mathbf{T}\cdot\frac{\partial \mathbf{u}}{\partial x} + \mathbf{D}\cdot\mathbf{n}E_x\right)\mathrm{d}s \tag{58}$$

where Γ is a curve surrounding the rigid line tip (starting counterclockwise from the lower surface and ending on the upper surface of the rigid line), H is the electric enthalpy density, $\mathbf{T}$ is the surface traction vector on Γ, $\mathbf{u}$ is the displacement vector, $\mathbf{D}$ is the electric displacement vector, $\mathbf{n}$ is the outward unit normal vector and s is the arc length. Since the integral is path-independent, we can take Γ as a circle with its center at the rigid line tip. After some calculation, we obtain

(1) for the conductive rigid line,

$$J_c = \frac{c\pi}{2\left[c_{44}^1\kappa_{11}^1 + \left(e_{15}^1\right)^2\right]}\left(c_{44}^1 D_c^2 - 2e_{15}^1 D_c\sigma_c - \kappa_{11}^1\sigma_c^2\right) \tag{59}$$

where

$$\begin{bmatrix} D_c \\ \sigma_c \end{bmatrix} = 2R^2\begin{bmatrix} \beta_1 & -1 \\ 1 & \alpha_1 \end{bmatrix}\left\{AR^2\mathbf{I} + \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}\right\}^{-1}\begin{bmatrix} 1+\mu_2 & \alpha_1+\alpha_2\mu_2 \\ \beta_1+\beta_2\mu_1 & -1-\mu_1 \end{bmatrix} \times\begin{bmatrix} 1 & 0 \\ \beta_2 & \left(1+\alpha_2\beta_2\right)\kappa_{11}^2 \end{bmatrix}\begin{bmatrix} \sigma_{xz}^0 \\ E_x^0 \end{bmatrix} \tag{60}$$

(2) for the insulating rigid line,

$$J_i = \frac{c\pi}{2\left[c_{44}^1\kappa_{11}^1 + \left(e_{15}^1\right)^2\right]}\left[c_{44}^1\left(D_1^2 - D_2^2\right) - 2e_{15}^1\left(D_1\sigma_1 - D_2\sigma_2\right) - \kappa_{11}^1\left(\sigma_1^2 - \sigma_2^2\right)\right] \tag{61}$$

where

$$\begin{bmatrix} D_1 + iD_2 \\ \sigma_1 + i\sigma_2 \end{bmatrix} = \frac{1}{Rc}\begin{bmatrix} \beta_1 & -1 \\ 1 & \alpha_1 \end{bmatrix}\left\{R^2\left[AR^2\mathbf{I} + \begin{pmatrix} A_1 & A_2 \\ A_3' & A_4' \end{pmatrix}\right]^{-1}\begin{bmatrix} B_1 \\ B_2' \end{bmatrix} + \frac{1}{A}\begin{bmatrix} \overline{B_1} \\ \overline{B_2} \end{bmatrix} - \frac{1}{A}\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}\left[AR^2\mathbf{I} + \begin{pmatrix} A_1 & A_2 \\ A_3' & A_4' \end{pmatrix}\right]^{-1}\begin{bmatrix} \overline{B_1} \\ \overline{B}_2' \end{bmatrix}\right\} \tag{62}$$

From eqns (59)–(62), it can be found that the extension force on the rigid line is less than or equal to zero when the applied shear stress ($\sigma_{xz}^0, \sigma_{yz}^0$) and electric field ($E_x^0, E_y^0$) satisfy some relations. This means that the driving force on the rigid line favors a contraction rather than an extension in the length of the rigid line.

When the inhomogeneity and matrix have the same elastic, piezoelectric and dielectric constants, eqns (59) and (61) can be simplified as

$$J_c = \frac{c\pi}{2c_{44}}\left[-\left(\sigma_{xz}^0\right)^2 + \left(c_{44}\kappa_{11} + (e_{15})^2\right)\left(E_x^0\right)^2\right] \tag{63}$$

for the conductive rigid line;

$$J_i = -\frac{c\pi}{2c_{44}\kappa_{11}}\left\{c_{44}\left[\beta\sigma_{yz}^0 + (1+\alpha\beta)\kappa_{11}E_y^0\right]^2 + \kappa_{11}(1+\alpha\beta)\left(\sigma_{xz}^0 + \alpha\kappa_{11}E_x^0\right)^2\right\} \tag{64}$$

for the insulating rigid line. Evidently, when the applied shear stress σ_{xz}^0 and electric field E_x^0 satisfy the following relation

$$\left(\frac{\sigma_{xz}^0}{E_x^0}\right)^2 \geqslant c_{44}\kappa_{11} + (e_{15})^2 \tag{65}$$

the extension force J_c on the conductive rigid line is less than or equal to zero. This implies that the driving force on the conductive rigid line favors a contraction in the length of the segment. This phenomenon can be found in the elastic problem (see Wang et al., 1986). For the insulating rigid line, the corresponding driving force is always negative. This means that there is a contraction near the insulating rigid line.

When the piezoelectric coupling effect is ignored, we have in terms of eqns (50) and (59)–(62)

$$J_c = \frac{2R^4 c\pi}{c_{44}^1 \kappa_{11}^1} \left\{ c_{44}^1 \frac{\left(\kappa_{11}^2 E_x^0\right)^2}{\left[(1+\mu_2)R^2 + 1 - \mu_2\right]^2} - \kappa_{11}^1 \frac{\left(\sigma_{xz}^0\right)^2}{\left[(1+\mu_1)R^2 + 1 - \mu_1\right]^2} \right\} \tag{66}$$

for the conductive rigid line;

$$J_i = -\frac{2R^4 c\pi}{c_{44}^1 \kappa_{11}^1} \left\{ c_{44}^1 \frac{\left(\kappa_{11}^2 E_y^0\right)^2}{\left[(1+\mu_2)R^2 - 1 + \mu_2\right]^2} + \kappa_{11}^1 \frac{\left(\sigma_{xz}^0\right)^2}{\left[(1+\mu_1)R^2 + 1 - \mu_1\right]^2} \right\} \tag{67}$$

for the insulating rigid line. From eqn (66), we can find that the applied shear stress σ_{xz}^0 has the decreasing effect on the extension force J_c, whereas the electric field E_x^0 has the increasing effect on it. This conclusion also is correct for eqn (63). Eqn (67) has an identical meaning with eqn (64) since J_i is negative. From eqns (64) and (67), it can be found that the present result can degenerate into one of Wu et al. (1998) when $e_{15}^1 = e_{15}^2 = 0$, $E_y^0 = 0$.

It should be shown that the individual contributions of mechanical and electric loadings need to be considered, respectively, since it is important to understand the character of piezoelectric media. For eqns (63) and (64), when the applied electric fields are equal to zeros, we can find that the extension forces J_c and J_i on the rigid line are negative and J_c which only depends on the shear modulus can degenerate into the result of Wang et al. (1986). However, from eqn (64), it can be seen that J_i is also related to piezoelectric and dielectric constants, except for the shear modulus. This shows that the path-independent integrals represent the different characters for the conductive and insulating rigid lines.

For eqns (66) and (67), it can be seen that when the piezoelectric coupling and applied electric fields are absent, these two equations become consistent. Here, the extension force on the rigid line only is related to shear moduli and can degenerate into the result of Wu et al. (1998).

5. Conclusions

1. The technique of conformal mapping and the theorem of analytic continuation are used to investigate the elastic field and electric field of a rigid line embedded in a confocal elliptic piezoelectric inhomogeneity in an infinite piezoelectric medium. The analytical solution to the elastic field and electric field is obtained.
2. The singularity behaviors of the stress field and electric displacement field at the rigid line tip are investigated. Some degenerate cases are taken into account. From eqns (54) and (55), it can be seen that only the applied shear stress σ_{xz}^0 and electric field E_x^0 have some influence upon the singularity of the shear stress and electric displacement fields when the rigid line is conductive. For the insulating rigid line, the shear stresses (σ_{xz}^0, σ_{yz}^0) and electric fields (E_x^0, E_y^0) have the effect on the singularity of the shear stress and electric displacement fields.
3. The stress and electric displacement singularity coefficients are derived. Two specific examples are provided. From eqns (56) and (57), it can be found that the stress and electric displacement singularity coefficients depend on the material constants, the geometric parameters of the elliptic inhomogeneity and the applied shear stress and electric fields at infinity.

4. The path-independent integral or the extension force on the rigid line is derived. From eqns (59)–(67), we can find that the extension force on the rigid line is less than or equal to zero when the applied shear stress and electric fields satisfy some conditions. This means that the driving force on the rigid line sometimes favors a contraction in the length of the segment.

Acknowledgements

This work was supported by the National Natural Science Foundation of China under the Grant of 19872023, the State Education Commission Foundation and the Failure Mechanics Laboratory of SEdC.

Appendix

It is well known that when the determinant of a second-order matrix is nonzero, this matrix is inverse. According to eqns (15), (37) and (39), we have after some manipulation

$$\begin{vmatrix} A - A_1 & -A_2 \\ -A_3 & A - A_4 \end{vmatrix} = 4\mu_1\mu_2(1 + \alpha_2\beta_2)\left[(1 + \mu_1)(1 + \mu_2) + \frac{(e_{15}^1 + e_{15}^2)^2}{c_{44}^1\kappa_{11}^1}\right] \tag{A1}$$

Since c_{44}^i and $\kappa_{11}^i (i = 1, 2)$ are positive real numbers, the above determinant also is positive according to eqn (15). Thus, we prove that the second inverse matrix on the right side of eqn (39) exists.

Subsequently, we will prove that the first matrix on the right side of eqn (39) is inverse. Its determinant can be expressed as

$$\begin{vmatrix} R^2A - A_1 & -A_2 \\ -A_3 & R^2A - A_4 \end{vmatrix} = (R^2A)^2 - (A_1 + A_4)R^2A + A_1A_4 - A_2A_3 \tag{A2}$$

The discriminant of this quadratic form can be written in terms of eqn (37) as

$$\Delta = 4(\mu_2 - \mu_1)^2 + 16\mu_1\mu_2(\alpha_1 - \alpha_2)(\beta_2 - \beta_1) \tag{A3}$$

If $\Delta < 0$, it is clear that the right side of eqn (A2) is positive. If $\Delta \geqslant 0$, we have

$$\alpha_1\beta_2\mu_1(\mu_1 + \mu_2) \geqslant (\alpha_1\beta_1 + \alpha_2\beta_2)\mu_1\mu_2 - \frac{(\mu_2 - \mu_1)^2}{4} \tag{A4}$$

According to eqns (9), (15), (A1) and (A4) and using the relation $\alpha_1\beta_2\mu_1 = \alpha_2\beta_1\mu_2$, we can obtain

$$\begin{vmatrix} R^2A - A_1 & -A_2 \\ -A_3 & R^2A - A_4 \end{vmatrix}$$

$$= (R^2 - 1)^2A^2 + 2(R^2 - 1)A\left(A - \frac{A_1 + A_4}{2}\right) + (A - A_1)(A - A_4) - A_2A_3$$

$$> 2(R^2 - 1)A[\mu_1 + \mu_2 + 2\mu_1\mu_2 + 2\alpha_1\beta_2\mu_1 + 2\alpha_2\beta_2\mu_1\mu_2]$$

$$> 2(R^2 - 1)A\left[\mu_1 + \mu_2 + 2\mu_1\mu_2 - \frac{(\mu_2 - \mu_1)^2}{2(\mu_1 + \mu_2)}\right]$$

$$> 0 \tag{A5}$$

Thus, we also prove that the first inverse matrix on the right side of eqn (39) exists.

References

Chung, M.Y., Ting, T.C.T., 1996. Piezoelectric solid with an elliptic inclusion or hole. International Journal of Solids and Structures 33, 3343–3361.

Deeg, W.F., 1980. The analysis of dislocation, crack, and inclusion problems in piezoelectric solids. Ph.D. thesis, Stanford University, Stanford, CA.

Kogan, L., Hui, C.Y., Molkov, V., 1996. Stress and induction field of a spheroidal inclusion or a penny-shaped crack in a transversely isotropic piezoelectric material. International Journal of Solids and Structures 33, 2719–2737.

Kuo, C.-M., Barnett, D.M., 1991. In modern theory of anisotropic elasticity and applications. In: Wu, J.J., Ting, T.C.T., Barnett, D.M. (Eds.), SIAM Proceedings, pp. 35–50.

Lee, J.S., Jiang, L.Z., 1996. Exact electroelastic analysis of piezoelectric laminae via state space approach. International Journal of Solids and Structures 33, 977–990.

Lothe, J., Barnett, D.M., 1976. Integral formalism for surface waves in piezoelectric crystal. Existence consideration. Journal of Applied Physics 47, 1799–1807.

Pak, Y.E., 1990. Crack extension force in a piezoelectric material. Journal of Applied Mechanics 57, 647–653.

Pak, Y.E., 1992. Circular inclusion problem in antiplane piezoelectricity. International Journal of Solids and Structures 29, 2403–2419.

Parton, V.Z., 1976. Fracture mechanics of piezoelectric materials. Acta Astro. 3, 671–683.

Rice, J.R., 1968. A path independent integral and the approximate analysis of strain concentration by notches and cracks. Journal of Applied Mechanics 35, 379–386.

Rice, J.R., 1969. Mathematical analysis in the mechanics of fracture. In: Liebowitz, H. (Ed.), Fracture—An Advanced Treatise, II. Academic Press, New York, pp. 191–311.

Sosa, H.A., Khutoryansky, N., 1996. New developments concerning piezoelectric materials with defects. International Journal of Solids and Structures 33, 3399–3414.

Sosa, H.A., Pak, Y.E., 1990. Three-dimensional eigenfunction analysis of a crack in a piezoelectric material. International Journal of Solids and Structures 26, 1–15.

Suo, Z., Kuo, C.-M., Barnett, D.M., Willis, J.R., 1992. Fracture mechanics for piezoelectric ceramics. Journal of Mechanics and Physics of Solids 40, 739–765.

Wang, Z.Y., Zhang, H.T., Chou, Y.T., 1985. Characteristics of the elastic field of a rigid line inhomogeneity. Journal of Applied Mechanics 52, 818–822.

Wang, Z.Y., Zhang, H.T., Chou, Y.T., 1986. Stress singularity at the tip of a rigid line inhomogeneity under antiplane shear loading. Journal of Applied Mechanics 53, 459–461.

Wu, L.Z., Meng, S.H., Du, S.Y., 1998. A rigid line in a confocal elliptic inhomogeneity embedded in an infinite medium under antiplane shear. Acta Mechanica Sinica 14, 266–273.

Zhang, T.-Y., Tong, P., 1996. Fracture mechanics for a mode II crack in a piezoelectric material. International Journal of Solids and Structures 33, 343–359.